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THE REALIZATION OF UNILATERAL CONSTRAINTS†

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The problem of realizing a one-sided constraint by means of an elastic force is considered. A limit theorem is established for more general assumptions on the non-potential generalized forces than in [1].

The general theorem on the realization of two-sided constraints by means of elastic forces was proposed by Courant and proved in [2]. An analogous theorem for one-sided constraints was stated in [1].

1. INITIAL EQUATIONS

Let a natural mechanical system be given in $\mathbf{R}^n = \{r\}$, subject to an ideal one-sided holonomic constraint defining a half-space M in $\mathbf{R}^n$ with boundary ∂M of dimensions $n_0 = n - 1$. Let $E(r, r^\cdot)$ be the kinetic energy of the system without constraints and let $F(r, r^\cdot)$ be the generalized active force. In a neighbourhood of any point on the manifold ∂M one can introduce coordinates $q \in \mathbf{R}$ and $r \in \mathbf{R}^{n_0}$ such that M is defined by the inequality $q \geqslant 0$ (and ∂M by $q = 0$) and the quadratic form E does not contain the product of $x^\cdot$ and $q^\cdot$. Therefore, henceforth we shall assume for simplicity that such coordinates are global, i.e. q is the first and x the remaining $n - 1$ components of r.

Then

$$E(r, r^\cdot) = \mathrm{T}(x, x^\cdot) + {}^1/_2 q^\cdot A(x) q^\cdot + O(|q|), \quad A(x) > 0 \tag{1.1}$$

The equations of motion have the form

$$(\partial E/\partial r^\cdot)^\cdot - \partial E/\partial r = F + R, \quad q \geqslant 0 \tag{1.2}$$

where R is the reaction of the constraint. The system moves under the constraint if $q = 0$ during the motion.

Consider the realization of a one-sided constraint by means of a force with potential NW, where N is a large positive parameter and

$$W = {}^1/_2 q B(x) q + O(|q|^3) \quad \text{for} \quad q < 0; \quad W = 0 \quad \text{for} \quad q \geqslant 0 \tag{1.3}$$

Henceforth we shall assume for simplicity that $B(x)$ is the same as the corresponding coefficient in the quadratic form $E(r, r^\cdot)$, i.e. $B(x) = A(x)$. The equations of motion of the system without constraints have the form

$$(\partial E/\partial r^\cdot)^\cdot - \partial E/\partial r = F - N\partial W/\partial r \tag{1.4}$$

2. REALIZATION OF THE MOTION OF THE SYSTEM WITH THE CONSTRAINT

Let $r_\infty(t)$ $(0 \leqslant t \leqslant \tau)$ be the motion of the system with a one-sided constraint given by (1.2), and kinetic energy E of the form (1.1), $R_\infty(t)$ being the reaction. Suppose that the following conditions are satisfied: the trajectory of motion belongs to ∂M, i.e. $q_\infty(t) = 0$ and $R_\infty(t) > 0$ for $0 \leqslant t \leqslant \tau$, and W has the form (1.3).

Let $r_N(t)$ be the motion (1.4) of the system with no constraint, given the initial conditions $r_N(0) = r_\infty(0)$ and $r_N^\cdot(0) = r_\infty^\cdot(0)$.

Theorem 1. For any sufficiently large N the motion is defined for $0 \leqslant t \leqslant \tau$ and

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$$r_N(t) = r_\infty(t) + O(N^{-1})^{\cdot}, \quad r_N^{\cdot}(t) = r_\infty^{\cdot}(t) + O(N^{-1/2}) \tag{2.1}$$

Remark. The estimate (2.1) can be refined

$$x_N^{\cdot}(t) = x_\infty^{\cdot}(t) + O(N^{-1})^{\cdot}, \quad q_N^{\cdot}(t) = q_\infty^{\cdot}(t) + O(N^{-1/2}) \tag{2.2}$$

3. AUXILIARY PROPOSITIONS

Proposition 1. Consider the initial conditions for (1.2) in a compact set G in the phase space $\mathbf{R}^{2n}$. Any solution $r(t)$ such that $r(0)$, and $r^{\cdot}(0)$ belong to G (with $q^{\cdot}(0)$, $q_0 \geqslant 0$) will then move away from the initial conditions by no more than $DN^{-1/2}$ during the time interval $\Delta t \leqslant \tau N^{-1/2}$ if N is sufficiently large. Moreover

$$D = C\tau + O(N^{-1/2}), \quad C = \text{const} \geqslant 0 \tag{3.1}$$

Proposition 2. Let the initial conditions for (1.4) belong to G with $-QN^{-1/2} \leqslant q_N^{\cdot}(0) \leqslant 0$ and $q_N(0) = 0$. Then for sufficiently large N

$$|x_N^{\cdot} - x_\infty^{\cdot}| \leqslant DN^{-1}, \quad |q_N - q_\infty| \leqslant DN^{-1}, \quad |q_N^{\cdot} - q_\infty^{\cdot}| \leqslant DN^{-1/2} \tag{3.2}$$

as long as $q_N \leqslant 0$ where r_∞, $r_\infty^{\cdot}$ is the solution of (1.2) with initial conditions $r_\infty(0) = r_N(0)$, $x_\infty^{\cdot}(0) = x_N^{\cdot}(0)$ and $q_\infty^{\cdot}(0) = 0$.

Proposition 2 is a direct consequence of a theorem in [3], according to which (2.1) and (2.2) are satisfied in the case of the realization of an ideal two-sided holonomic constraint with the aid of a force with potential NW ($W(r)$ reaches a minimum on the constraint manifold). The estimates (2.2) remain valid if the initial condition $q_\infty^{\cdot}(0)$ is replaced by $O(N^{-1/2})$, which follows from [3].

4. PROOF OF THEOREM 1

In the phase space $\mathbf{R}^{2n}$ we consider a domain G which is a neighbourhood of the solution r_∞, $r_\infty^{\cdot}$. Let F_g be the projection of the generalized force F onto the direction of q. Then

$$-m \geqslant F_q + \partial E/\partial q \geqslant -M, \quad M > m > 0 \tag{4.1}$$

in G.

The kinetic energy $E(r, r^{\cdot})$ has the form (1.1) with $a \leqslant A(x) \leqslant A$; and $W(r)$ has the form (1.3). The equality

$$(\partial E/\partial q^{\cdot})^{\cdot} - q^{\cdot\cdot}A(x) - (\partial A(x)/\partial x)x^{\cdot}q^{\cdot} = O(q) + O(q^{\cdot}) \tag{4.2}$$

holds and the $O(\cdot)$ functions on the right-hand side are uniformly bounded in G.

Consider the motion of the free system. Since $q_N(0) = q_N^{\cdot}(0) = 0$ and $q_N^{\cdot\cdot}(0) < 0$, it is seen from (4.1) that $q_N(0)$ becomes negative at the beginning of the motion and the estimates (2.1) hold. Suppose that the trajectory of the system lies in the half-space $q > 0$, i.e. q_N is equal to zero and $q_N^{\cdot}$ is positive at a certain instant t_0. We call this a "jump". Then, by (2.1)

$$q_N^{\cdot}(t_0) \leqslant DN^{-1/2}, \quad |x_N(t_0) - x_\infty(t_0)| \leqslant DN^{-1}, \quad |x_N^{\cdot}(t_0) - x_\infty^{\cdot}(t_0)| \leqslant DN^{-1}$$

It can be shown that the time during which the system moves "above" the constraint is bounded. Indeed, since there is a time s, greater than t_0, such that $q_N^{\cdot}(s) = 0$, we have

$$q_N^{\cdot}(s) = q_N^{\cdot}(t_0) + \int_{t_0}^{s} q_N^{\cdot\cdot}(\xi)d\xi$$

It follows that

$$\int_{t_0}^{s} q_N^{\cdot\cdot}(\xi)d\xi \geqslant -2DN^{-1/2}$$

$$\ddot{q}_N = [F_q + \partial E/\partial q - (\partial A/\partial x)\dot{x}\dot{q}_N + O(q_N) + O(\dot{q}_N)]/A(x)$$

This means that $|\ddot{q}_N(t)| \leqslant m/(2A)$ for sufficiently large N. Therefore

$$s - t_0 \leqslant S/2, \quad S = 8DA/(mN^{1/2}) \tag{4.3}$$

By analogy, it can be shown that if t_1 is a time such that $q_N(t_1) = 0$ and $\dot{q}_N(t_1) < 0$, then

$$t_1 - s \leqslant S/2 \tag{4.4}$$

Since $|\ddot{q}_N(t)| \leqslant 2M/a$ for $t_0 \leqslant t \leqslant t_1$, then at least

$$|\dot{q}_N(t)| < MS/(2a) \tag{4.5}$$

The difference $|\dot{q}_N(t_0)| - |\dot{q}_N(t_1)|$ can be estimated as follows:

$$|\dot{q}_N(t_0) + \dot{q}_N(t_1)| = \int_{t_0}^{s} \ddot{q}_N(\xi)d\xi - \int_{s}^{t_1} \ddot{q}_N(\xi)d\xi$$

$$(\dot{q}_N(t_0)) > 0, \quad \dot{q}_N(t_1) < 0)$$

Inequality (4.3) implies that a constant C exists such that

$$|\ddot{q}_N| \leqslant |(F_q + \partial E/\partial q)/A(x)| + C|\dot{q}_N|$$

since $|((\partial A/\partial x)\dot{x} - O(1))/A(x)|$ is a function uniformly bounded in G.

For positive values of q the motion of the free system can be described by the same equations as the motion of the system with one-sided constraint. It follows that the solution shifts by no more than $N^{-1/2}$ during a time S, where k is proportional to D within $O(N^{-1/2})$ (see (3.1)). As has been demonstrated, (4.5) holds. Therefore $\ddot{q}_N$ changes by no more than $QN^{-1/2}$. Then

$$|\dot{q}_N(t_0) + \dot{q}_N(t_1)| \leqslant SQN^{-1/2} = C_1 D^2 N^{-1}$$

We will now consider entering the region "under the constraint"

$$q_N(t_1) = 0, \quad \dot{q}_N(t_1) = -D_1 N^{-1/2}, \quad D_1 < 2D$$

Then

$$\ddot{q}_N + Nq_N = \frac{1}{A(x)}\left[F_q + \frac{\partial E}{\partial q} - \frac{\partial A}{\partial x}\dot{x}\dot{q}_N + O(q_N) + O(\dot{q}_N)\right] \tag{4.6}$$

By (3.2), there is a positive constant K such that

$$|\dot{q}_N| \leqslant KN^{-1/2}, \quad |r_N - r_\infty| \leqslant KN^{-1}, \quad |\dot{x}_N - \dot{x}_\infty| \leqslant KN^{-1}$$

After a time $t < 2\pi N^{-1/2}$ the right-hand side of (4.6) changes by no more than $Q_1 N^{-1/2}$. It follows that

$$\ddot{q}_N = -Nq_N + F_0 \pm Q_1 N^{-1/2}, \quad F_0 = \frac{1}{A(x)}\left(F_q + \frac{\partial E}{\partial q}\right)\Bigg|_{t=t_1}$$

and q_N can be given with accuracy up to $\varepsilon = Q_2 N^{-3/2}$

$$q_N \pm \varepsilon = -\frac{D_1}{N}\sin((N^{1/2}(t - t_1)) - \frac{F_0}{N}\cos(N^{1/2}(t - t_1)) + \frac{F_0}{N} \tag{4.7}$$

The right-hand side of (4.7) is equal to zero when

$$t = t_1, \quad t = t_2; \quad t_2 = t + N^{-1/2} \arcsin\frac{2D_1F_0}{D_1^2 + F_0^2}$$

Consequently, $q_N = 0$ for $t = t_1$ and $t = t_2 \pm S/N$, and the upper estimate for S is independent of D_1 (as D_1 increases Q_2 remains unchanged, and so S decreases).

It follows that $|q_N^{\cdot}(t_1) + q_N^{\cdot}(t_2)| < C_2N^{-1}$.

At the next "jump"

$$|q_N^{\cdot}| < 2DN^{-1/2} \tag{4.8}$$

The whole of the preceding discussion therefore holds at least as long as $(C_1D^2N^{-1} + C_2N^{-1})K < DN^{-1/2}$ (K is the number of "jumps").

Let Δt be the time of a "jump". Then $K = T_1/\Delta t \leqslant T_1N^{-1/2}/(C_3D)$, where T_1 is the time interval during which (4.8) is satisfied

$$T_1 \geqslant C_3D^2/(C_1D^2 + C_2)$$

At $t = T_1$ we have $|q^{\cdot}(t)| < 3DN^{-1/2}$ and the whole discussion can be repeated with D replaced by $\frac{3}{2}D$. The time interval T_2 during which $q_N^{\cdot}(t) < 3DN^{-1/2}$ will then be longer than $C_3(2D)/(C_1(2D)^2 + C_2)$.

Let T_n be a time interval such that

$$(n-1)DN^{-1/2} < q_N^{\cdot}(t) < nDN^{-1/2}.$$

Then the sum $T_1 + \ldots + T_n$ can be made a large as required: $T_1 + \ldots + T_{n-1} \leqslant \tau < T_1 + \ldots + T_n$. It follows that for $0 \leqslant t \leqslant \tau$

$$q_N(t) = Q(N^{-1}), \quad q_N^{\cdot}(t) = O(N^{-1/2}).$$

Now, it can be shown that

$$x_N(t) = x_\infty(t) + O(N^{-1}), \quad x_N^{\cdot}(t) = x_\infty^{\cdot}(t) + O(N^{-1}) \tag{4.9}$$

Let $y = \partial E/\partial x^{\cdot}$. Then

$$x_\infty^{\cdot} = \partial E/\partial y, \quad y_\infty^{\cdot} = -\partial E/\partial x + F(x) \tag{4.10}$$

(since the constraint is ideal, the project of the reaction R onto any of the x directions is equal to zero). We use the equations

$$x_N^{\cdot} = \partial E/\partial y + O(N^{-1}), \quad y_N^{\cdot} = -\partial E/\partial x + F(x) + O(N^{-1})$$

Because x_N and y_N satisfy (4.10) to within $O(N^{-1})$, we obtain (4.9) by the smoothness of all the functions.

5. LEAVING THE CONSTRAINT

Let $r_\infty(t)$ be the motion of the system with one-sided constraint given by (1.2) with kinetic energy E of the form (1.1), let R_∞ be the reaction of the constraint, and let $0 \leqslant t \leqslant \tau$.

Let the system move on the constraint for $0 \leqslant t < \tau_{cx}$, i.e. $q_\infty(t) = 0$ and $R_\infty > 0$, leaving the constraint at $t = \tau_{cx}$, and suppose a positive constant δ exists such that $q_\infty > 0$ for $t \in (\tau_{cx}, \tau cx + \delta]$.

Let $r_N(t)$ be the motion (1.4) of the system without a constraint, with W of the form (1.3) and $r_N(0) = r_\infty(0)$, $r_N^{\cdot}(0) = r_\infty^{\cdot}(0)$.

Theorem 2. For any sufficiently large N the motion is defined for $0 \leqslant t < \tau_{cx} + \delta$ and the equalities

$$r_N = r_\infty + O(N^{-1/2}), \quad \dot{r_N} = \dot{r_\infty} + O(N^{-1/2}) \tag{5.1}$$

are satisfied.

Remarks. Equalities (2.1) and (2.2) are satisfied when system (1.4) moves on the constraint, i.e. when $t \leqslant \tau_\alpha$. When $t > \tau_\alpha + \delta$ system (1.2) can reach the surface $q = 0$ again (either smoothly or with an impact). However, Theorem 2 does not cover these questions (see [2]).

6. PROOF OF THEOREM 2

In the phase space $\mathbf{R}^{2n}$ let G be a neighbourhood of the solution r_∞, $\dot{r_\infty}$ and let F_q be the project of F onto the direction of q.

Since all the functions are assumed to be multiply differentiable, there is a time τ_1 such that $|F_q + \partial E/\partial q|$ decreases monotonically for $t > \tau_1$. By Theorem 1, the motion is defined for $t \leqslant \tau_1$ and the equalities (2.1) and (2.2) are satisfied.

Suppose that the system turns out to be "above" the constraint for $t > \tau_1$ with "exit" velocity $\dot{q_N} = DN^{-1/2}$. Then, since $|F_q < \partial E/\partial q|$ is monotonically decreasing, the modulus of $\dot{q_N}$ at the time when $q_N = 0$ and $\dot{q_N} < 0$ does not exceed that of $\dot{q_N}$ at the time when $q_N = 0$ and $\dot{q_N} < 0$. In the half-space $\{q < 0\}$ the coordinate q_N has the form (4.7). Differentiating with respect to t and substituting $t = t_2 + S/N$, we obtain

$$|\dot{q_N}(t_1) + \dot{q_N}(t)| \leqslant 2F_0S/N$$

Therefore, at each "jump" the modulus of $\dot{q_N}$ increases by no more than $2F_0S/N$. Since the time of a "jump" is not less than $CDN^{-1/2}$, the time interval T_1 during which (4.8) is satisfied is longer than $D^2/(2SF_0)$. It follows that the motion is defined for $0 \leqslant t \leqslant \tau_\alpha$ and the equalities (2.1) and (2.2) are satisfied. For $t \leqslant \tau_\alpha$ there is a time $t_\alpha = \tau_\alpha + O(N^{-1/2})$ such that $q_N(t_\alpha) = 0$. Within the time interval $[t_\alpha, \tau_\alpha + \delta]$ the system with one-sided constraint and the free system are described by the same equations if N is sufficiently large. Therefore, since the estimates (3.1) hold when $t = t_\alpha$ (and so r_N and $\dot{r_N}$ differ from r_∞ and, respectively, $\dot{r_\infty}$ by $O(N^{-1/2})$ at this instant), the theorem on the continuous dependence of the solution on the initial conditions implies that the estimates (5.1) are satisfied in this interval. The theorem is proved.

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